Philosophy 159 - Homework 8

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2003 December 3

17.2 Proposition 1: The sentence S is a tautological consequence of the set $\mathcal{T}$ if and only if the set $\mathcal{T} \cup \{\neg\mathsf{S}\}$ is not tt-satisfiable.

($\Rightarrow$) Assume that the sentence S is a tautological consequence of the set $\mathcal{T}$. This means that every truth assignment which makes all of the sentences in $\mathcal{T}$ true also makes S true. Looking for a contradiction, assume that the set $\mathcal{T} \cup \{\neg\mathsf{S}\}$ is tt-satisfiable. This means that there is a single truth assignment which makes all of the sentences in $\mathcal{T} \cup \{\neg\mathsf{S}\}$ true. This assignment makes $\neg\mathsf{S}$ true. It also all of the sentences in $\mathcal{T}$ true, so by assumption that S is a tautological consequence of $\mathcal{T}$, this also makes S true. So we have both S and $\neg\mathsf{S}$, which gives us a contradiction. Thus, $\mathcal{T} \cup \{\neg\mathsf{S}\}$ is not tt-satisfiable.

($\Leftarrow$) Assume that the set $\mathcal{T} \cup \{\neg\mathsf{S}\}$ is not tt-satisfiable. Thus, there is no truth assignment which makes all of the sentences in $\mathcal{T} \cup \{\neg\mathsf{S}\}$ true. Any truth assignment h must make at least one of the sentences in $\mathcal{T} \cup \{\neg\mathsf{S}\}$ false. If we let h be any truth assignment which makes all of the sentences in $\mathcal{T}$ true, it must therefore make $\neg\mathsf{S}$ false, making S true. Thus, any truth assignment which makes all of the sentences in $\mathcal{T}$ true also makes S true, which means that S is a tautological consequence of $\mathcal{T}$.

17.3 h_1 and h_2 are truth assignments which agree on all the atomic sentences in S. We want to show that $\hat{h}_1(\mathsf{S}) = \hat{h}_2(\mathsf{S})$. Prove this by induction on wffs (as defined in exercise 16.12).

Base Case: If Q is an atomic sentence, then part (1) of the definition of $\hat{h}$ tells us that $\hat{h}_1(\mathsf{Q}) = h_1(\mathsf{Q})$ and $\hat{h}_2(\mathsf{Q}) = h_2(\mathsf{Q})$. We know that h_1 and h_2 agree on Q, so $\hat{h}_1$ and $\hat{h}_2$ also agree on Q.

Inductive Step: Assume that Q and R are wffs such that $\hat{h}_1(\mathsf{Q}) = \hat{h}_2(\mathsf{Q})$ and $\hat{h}_1(\mathsf{R}) = \hat{h}_2(\mathsf{R})$.

Part (2) of the definition of $\hat{h}$ tells us that $\hat{h}_1(\neg\mathsf{Q})$ depends only on (is exactly the opposite of) the value of $\hat{h}_1(\mathsf{Q})$, and similar for $\hat{h}_2(\neg\mathsf{Q})$. Because we know $\hat{h}_1(\mathsf{Q}) = \hat{h}_2(\mathsf{Q})$, this tells us that $\hat{h}_1(\neg\mathsf{Q}) = \hat{h}_2(\neg\mathsf{Q})$.

Part (3) of the definition of $\hat{h}$ tells us that $\hat{h}_1(\mathsf{Q} \wedge \mathsf{R})$ depends only on the values of $\hat{h}_1(\mathsf{Q})$ and $\hat{h}_1(\mathsf{R})$. Similar for $\hat{h}_2(\mathsf{Q} \wedge \mathsf{R})$. Because we know $\hat{h}_1(\mathsf{Q}) = \hat{h}_2(\mathsf{R})$ and $\hat{h}_1(\mathsf{Q}) = \hat{h}_2(\mathsf{R})$, this tells us that $\hat{h}_1(\mathsf{Q} \wedge \mathsf{R}) = \hat{h}_2(\mathsf{Q} \wedge \mathsf{R})$.

Parallel reasoning gives $\hat{h}_1(\mathsf{Q} \vee \mathsf{R}) = \hat{h}_2(\mathsf{Q} \vee \mathsf{R})$, $\hat{h}_1(\mathsf{Q} \rightarrow \mathsf{R}) = \hat{h}_2(\mathsf{Q} \rightarrow \mathsf{R})$ and $\hat{h}_1(\mathsf{Q} \leftrightarrow \mathsf{R}) = \hat{h}_2(\mathsf{Q} \leftrightarrow \mathsf{R})$.

Because all wffs are built up according to the definition in this manner, this shows that $\hat{h}_1(\mathsf{S}) = \hat{h}_2(\mathsf{S})$ for any wff S.

17.5 $\mathcal{T}$: $\{\neg(\mathsf{Cube(a)} \vee \mathsf{Small(a)}), \mathsf{Cube(b)} \rightarrow \mathsf{Cube(a)}, \mathsf{Small(a)} \vee \mathsf{Small(b)}\}$

Formal Consistency: We want to show that $\mathcal{T} \nvdash_{\mathrm{T}} \bot$. Towards a proof by contradiction, assume $\mathcal{T} \vdash_{\mathrm{T}} \bot$. By the soundness theorem, this means that $\bot$ is a tautological consequence of $\mathcal{T}$. Thus, every truth assignment which makes all of the sentences in $\mathcal{T}$ true will also make $\bot$ true. However, we can find a truth assignment ($\mathsf{Cube(a)}$ = FALSE, $\mathsf{Cube(b)}$ = FALSE, $\mathsf{Small(a)}$ = FALSE, $\mathsf{Small(b)}$ = TRUE) which makes all of the sentences in $\mathcal{T}$ true without generating a contradiction. This contradicts our assumption that $\mathcal{T} \vdash_{\mathrm{T}} \bot$, and thus $\mathcal{T} \nvdash_{\mathrm{T}} \bot$.

Formal Completeness: By Lemma 5, we want to show that, for every atomic sentence A in our language, $\mathcal{T} \vdash_{\mathrm{T}} A$ or $\mathcal{T} \vdash_{\mathrm{T}} \neg A$. Our language includes two predicates and two constants, for a total of four atomic sentences: $\mathsf{Cube(a)}$, $\mathsf{Small(a)}$, $\mathsf{Cube(b)}$, and $\mathsf{Small(b)}$. The first sentence of $\mathcal{T}$ gives us $\neg\mathsf{Cube(a)}$ and $\neg\mathsf{Small(a)}$. The second sentence, combined with $\neg\mathsf{Cube(a)}$, proves $\neg\mathsf{Cube(b)}$. The final sentence, combined with $\neg\mathsf{Small(a)}$, proves $\mathsf{Small(b)}$.

17.6 $\mathcal{T}$: $\{\neg(\mathsf{Cube(a)} \lor \mathsf{Small(a)}), \mathsf{Cube(b)} \to \mathsf{Cube(a)}, \mathsf{Small(a)} \lor \mathsf{Small(b)}\}$

The truth assignment h which makes all of the sentences in $\mathcal{T}$ true assigns the following values to the atomic sentences of the language: $\mathsf{Cube(a)}$ = FALSE, $\mathsf{Cube(b)}$ = FALSE, $\mathsf{Small(a)}$ = FALSE, $\mathsf{Small(b)}$ = TRUE.

17.7 $\mathcal{T}$: $\{\neg(\mathsf{Cube(a)} \land \mathsf{Small(a)}), \mathsf{Cube(b)} \to \mathsf{Cube(a)}, \mathsf{Small(a)} \lor \mathsf{Small(b)}\}$

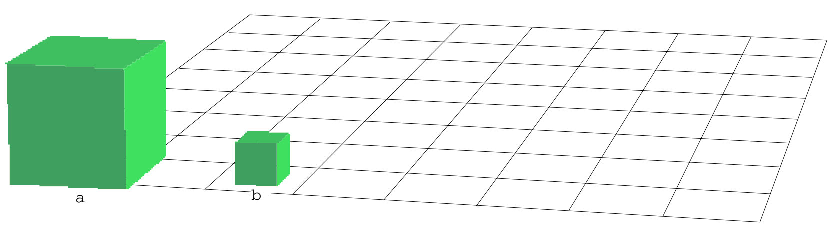
Alphabetical ordering of atomic sentences: $\mathsf{A}_1 = \mathsf{Cube(a)}$, $\mathsf{A}_2 = \mathsf{Cube(b)}$, $\mathsf{A}_3 = \mathsf{Small(a)}$, and $\mathsf{A}_4 = \mathsf{Small(b)}$.

- Neither $\mathsf{Cube(a)}$ nor $\neg\mathsf{Cube(a)}$ is provable from $\mathcal{T}$, so we add $\mathsf{Cube(a)}$ to the set.
- Neither $\mathsf{Cube(b)}$ nor $\neg\mathsf{Cube(b)}$ is provable from $\mathcal{T}$, so we add $\mathsf{Cube(b)}$ to the set.
- From $\neg(\mathsf{Cube(a)} \land \mathsf{Small(a)})$ and $\mathsf{Cube(a)}$, we can prove $\neg\mathsf{Small(a)}$.
- From $\mathsf{Small(a)} \lor \mathsf{Small(b)}$ and $\neg\mathsf{Small(a)}$, we can prove $\mathsf{Small(b)}$.

The expanded formally complete set is:
$\{\neg(\mathsf{Cube(a)} \land \mathsf{Small(a)}), \mathsf{Cube(b)} \to \mathsf{Cube(a)}, \mathsf{Small(a)} \lor \mathsf{Small(b)}, \mathsf{Cube(a)}, \mathsf{Cube(b)}\}$

The truth assignment h is such that: $h(\mathsf{Cube(a)})$ = TRUE, $h(\mathsf{Cube(b)})$ = TRUE, $h(\mathsf{Small(a)})$ = FALSE, and $h(\mathsf{Small(b)})$ = TRUE.

A world making all of the sentences in the formally complete set is shown below.



17.14 Lemma 3, Part 4: $\mathcal{T} \vdash_{\mathrm{T}} (\mathsf{R} \to \mathsf{S})$ iff $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$

($\Leftarrow$) Assume $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$. We have to show that, in either case, we can prove $\mathsf{R} \to \mathsf{S}$.

Assume $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$. Because $\mathcal{T}$ is formally complete, this means that $\mathcal{T} \vdash_{\mathrm{T}} \neg\mathsf{R}$. Suppose the proof of $\neg\mathsf{R}$ uses the premises $\mathsf{P}_1, \ldots, \mathsf{P}_n$ and looks like this:

$$
\begin{array}{|l}
\mathsf{P}_1 \\
\vdots \\
\mathsf{P}_n \\
\hline
\vdots \\
\neg\mathsf{R}
\end{array}
$$

We can form a proof of $\mathsf{R} \rightarrow \mathsf{S}$ as follows:

$$
\begin{array}{|lll}
\mathsf{P}_1 & & \\
\vdots & & \\
\mathsf{P}_n & & \\
\hline
\vdots & & \\
\neg\mathsf{R} & & \\
\quad \begin{array}{|l} \mathsf{R} \\ \hline \neg\mathsf{R} \\ \bot \\ \mathsf{S} \end{array} & \begin{array}{l} \\ \textbf{Reit} \\ \bot\ \textbf{Intro} \\ \bot\ \textbf{Elim} \end{array} \\
\mathsf{R} \rightarrow \mathsf{S} & \rightarrow \textbf{Intro}
\end{array}
$$

For the second case, assume $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$. Suppose the proof of S uses the premises $\mathsf{P}_1, \ldots, \mathsf{P}_n$ and looks like this:

$$
\begin{array}{|l}
\mathsf{P}_1 \\
\vdots \\
\mathsf{P}_n \\
\hline
\vdots \\
\mathsf{S}
\end{array}
$$

Then we can show $\mathsf{R} \rightarrow \mathsf{S}$ as follows:

$$
\begin{array}{|ll}
\mathsf{P}_1 & \\
\vdots & \\
\mathsf{P}_n & \\
\hline
\vdots & \\
\mathsf{S} & \\
\quad \begin{array}{|l} \mathsf{R} \\ \hline \mathsf{S} \end{array} & \begin{array}{l} \\ \textbf{Reit} \end{array} \\
\mathsf{R} \rightarrow \mathsf{S} & \rightarrow \textbf{Intro}
\end{array}
$$

17.15 Lemma 3, Part 4: $\mathcal{T} \vdash_{\mathrm{T}} (\mathsf{R} \rightarrow \mathsf{S})$ iff $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$

($\Rightarrow$) Assume $\mathcal{T} \vdash_{\mathrm{T}} (\mathsf{R} \rightarrow \mathsf{S})$. We need to show that either $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$. By Lemma 3, part 3, this result that we're trying to show is equivalent to $\mathcal{T} \vdash_{\mathrm{T}} \neg\mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$. By Lemma 3, part 2, this is equivalent to $\mathcal{T} \vdash_{\mathrm{T}} (\neg\mathsf{R} \vee \mathsf{S})$.

Toward a proof by contradiction, assume $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{R}$ and $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{S}$. By Lemma 3 part 3, this is equivalent to $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{R}$ and $\mathcal{T} \vdash_{\mathrm{T}} \neg\mathsf{S}$. By Lemma 3 part 1, this is equivalent to $\mathcal{T} \vdash_{\mathrm{T}} (\mathsf{R} \wedge \neg\mathsf{S})$. By DeMorgan's law, this is equivalent to $\mathcal{T} \vdash_{\mathrm{T}} \neg(\neg\mathsf{R} \vee \mathsf{S})$. Combining this proof with the proof of $(\neg\mathsf{R} \vee \mathsf{S})$ above, and adding one step of $\perp$ **Intro**, we get a contradiction. Thus our assumption, that $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{R}$ and $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{S}$ is false. This means that either $\mathcal{T} \nvdash_{\mathrm{T}} \mathsf{R}$ or $\mathcal{T} \vdash_{\mathrm{T}} \mathsf{S}$, which is what we were trying to show.